ON THE CONTINUITY AND LESCHE STABILITY OF TSALLIS AND RÉNYI ENTROPIES AND Q-EXPECTATION VALUES

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Abstract. It is shown that the Rényi and Tsallis entropies and the q-expectation values, are continuous and stable if $q > 1$ and are not continuous and instable for uniform finite distributions if $q < 1$.

1. Introduction

Experimental robustness is a natural criteria of physical quantities requiring that

> A physically meaningful function of a probability distribution should not change drastically if the underlying distribution function is slightly changed.

Lesche in 1982 has given a mathematical formulation of the above requirement, called stability, and proved that the entropy of Rényi is not stable [1]. Based on Lesche's reasoning later on Abe has shown that the Tsallis entropy is stable [2]. Lesche stability became a criteria in distinguishing and favoring one of the many different entropies in non-extensive thermostatistics [3, 4, 5, 6, 7, 8, 9] and the proofs of Lesche and Abe become one of the arguments in favoring Tsallis entropy to Rényi. Lesche stability as a proper concept of experimental robustness was questioned and attacked by several authors [10, 11, 12]. They have collected physical arguments claiming that Lesche stability do not express properly the physical content of experimental robustness. Lesche and Abe rejected these arguments [13, 14]. Recently Abe recognized that the central quantities of non-extensive statistical mechanics, the q-averages [15], are Lesche instable [16]. This important observation somehow invalidates the whole mathematical framework of non-extensive thermostatistics, therefore several authors argued again that Lesche stability is a too strict concept for physical applications and suggested different modifications [17, 18, 19].

The concept of experimental robustness is a lousy continuity requirement and enables several mathematical formulations. Considering this resemblance to continuity the instability of the Rényi entropy S_R (1) and the stability of Tsallis entropy S_T (2) is somehow paradoxical, because the Tsallis entropy $S_T = (1-e^{(1-q)S_R})/(q-1)$ (where $0 < q \neq 1$) is a smooth function of the Rényi entropy.

In the following we investigate some mathematical concepts releted to the Rényi and Tsallis entropies and q-expectation values. We introduce a local form of Lesche stability, that, according to our opinion, expresses best the physical content of experimental robustness.

Date: October 10, 2009.

2. Continuity of functions of probability distributions

The simplest formulation of experimental robustness is continuity. Recall the following notions.

The set of infinite discrete probability distributions is

$$D := \left\{p \in l^1 |\ \|p\|_1 = 1,\ p_i \geq 0,\ i \in \mathbb{N}\right\} \subset l^1.$$

Here the l^1 norm is used as the natural concept of distance [20].

Let X be a normed space with norm $\|\ \|$.

Definition 1: A function $f : D \to X$ is *continuous at* p if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall r)(\|r - p\|_1 < \delta \Rightarrow \|f(r) - f(p)\| < \epsilon).$$

f is *continuous* if it is continuous at every $p \in D$.

Note that if there is a positive number c_p so that $\|f(r) - f(p)\| < c_p\|r - p\|_1$ then f is continuous at p.

Definition 2: A function $f : D \to X$ is *uniformly continuous* if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall r, p)(\|r - p\|_1 < \delta \Rightarrow \|f(r) - f(p)\| < \epsilon).$$

Note that if there is a positive number c so that $\|f(r) - f(p)\| < c\|r - p\|_1$ then f is uniformly continuous.

Continuity is a *local* property while uniform continuity is a *global* property.

Observe that the negation of continuity reads as follows:

$$(\exists p)(\exists \epsilon > 0)(\forall \delta > 0)(\exists r, \|r - p\|_1 < \delta)(\|f(r) - f(p)\| \geq\ \epsilon)$$

and the negation of unformly continuity:

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists r, p, \|r - p\|_1 < \delta)(\|f(r) - f(p)\| \geq\ \epsilon).$$

2.1. **1<q.** The Banach space of real sequences for which of the corresponding series is convergent at the power q, is denoted by l^q, and the Banach space of bounded sequences is denoted by l^∞. We know that if $k \in l^1$ and $1 < q$, then $k \in l^q$ and $\|k\|_q \leq \|k\|_1$. Therefore the q-norm, as the function $\|.\|_q :\ l^1 \to \mathbb{R}, k \mapsto \|k\|_q$ function, is uniformly continuous.

Proposition 1 The function $D \to l^1,\ p \mapsto p^q := (p_i^q)_{i\in\mathbb{N}}$ is uniformly continuous.

Proof: According to the mean value theorem of differential calculus

$$\|r^q - p^q\|_1 = \sum_{i\in\mathbb{N}} |r_i^q - p_i^q| \leq \sum_{i\in\mathbb{N}} q|r_i - p_i| = q\|r - p\|_1.$$

∎

Note that $\|p^q\|_1 = \|p\|_q$.

Corollary 1.1 The Rényi entropy

$$S_R :\ D \to \mathbb{R}, \qquad p \mapsto \frac{1}{1-q} \ln \|p\|_q, \tag{1}$$

is continuous and the Tsallis entropy

$$S_T :\ D \to \mathbb{R}, \qquad p \mapsto \frac{1 - \|p\|_q}{q - 1} \tag{2}$$

if $1 < q$ is uniformly continuous. ∎

The *expectation value* of $A = (A_i)_{i\in\mathbb{N}} \in l^\infty$,

$$D \to \mathbb{R}, \qquad p \mapsto (A|p) = \sum_{i\in\mathbb{N}} A_i p_i,$$

is uniformly continuous.

In general, if $\Phi : D \to D$ is a given function, then the Φ-expectation value of A is

$$D \to \mathbb{R}, \qquad p \mapsto (A|\Phi(p)).$$

If Φ is (uniformly) continuous, then the Φ-expectation value is (uniformly) continuous.

Corollary 1.2 The q-expectation value, where $\Phi(p)_i := \frac{p_i^q}{\|p^q\|_1}$ (the quotient of continuous functions) is continuous.

2.2. **q<1.** In this case the summability of p^q for $p \in D$ is not automatic. Therefore the previous functions (entropies and q averages) are interpreted on the set:

$$D_q := \{p \in D| \;\; p^q \in l^1\}.$$

Proposition 2 The function $D_q \to l^1$, $p \mapsto p^q$ is not continuous at finite uniform distributions.

Proof: Let be $n \in \mathbb{N}$ a given number and

$$p := \left(\frac{1}{n}, ..., \frac{1}{n}, 0, 0, ...\right) \in D,$$

therefore the number of nonzero elements is n. In the following we will use the notation

$$p = \left(\frac{1}{n}\bigg|_{\times n}, 0 \right). \tag{3}$$

For all $0 < \delta < 1/2$ let us define

$$r_\delta := \left(\frac{1-\delta}{n}\bigg|_{\times n}, \frac{\delta}{m}\bigg|_{\times m}, 0 \right), \tag{4}$$

where

$$m \geq \delta^{\frac{q}{q-1}} \left(1 + q\delta n^{1-q}\right)^{\frac{1}{1-q}}.$$

Then $\|r_\delta - p\|_1 = 2\delta$, however

$$\| \, p_\delta^q - p^q \, \|_1 = ((1-\delta)^q - 1)n^{1-q} + \delta^q m^{1-q} \geq 1.$$

■

Since the logarithm and the identity are injective continuous functions, we have:

Corollary 2.1 The Rényi and Tsallis entropies, if $q < 1$, are not continuous.■

Note that the proof of the previous proposition is essentially identical that of Lesche in [13], regarding the instability of Rényi entropy. However, the above argumentation is not applicable in the case $1 < q$. In particular, it is impossible to determine m so that $m^{1-q} \geq \delta^{-q}(1 + (1-(1-\delta)^q)n^{1-q})$, because then the direction of the inequality is reversed by the negative powers

$$m^{q-1} \leq \frac{\delta^q}{(1 + (1-(1-\delta)^q)n^{1-q})} < \delta^q.$$

Hence, there is no $m \in \mathbb{N}$ that satisfies the inequality, if $\delta < 1$.

Let us know investigate the continuity of the expectation values. Here it is not enough to show that the function $p \mapsto \frac{p^q}{\|p^q\|}$ is not continuous, because the strong convergence (convergence in norm) does not follow from the weak convergence. What we show is that $p \mapsto (A|p^q/\|p^q\|_1)$ is not continuous for a large set of $A \in l^\infty$.

Let be p and p^q are chosen as previously. Then

$$\|p_\delta^q\|_1 = (1-\delta)^q n^{1-q} + \delta^q m^{1-q}. \tag{5}$$

and

$$\frac{p_\delta^q}{\|p_\delta^q\|_1} - \frac{p^q}{\|p^q\|_1} = \frac{\delta^q}{m^{q-1}\|p_\delta^q\|_1}\left(-\frac{1}{n}\bigg|_{\times n}, \frac{1}{m}\bigg|_{\times m}, 0\right).$$

Therefore

$$\left|\left(A\middle|, \frac{p_\delta^q}{\|p_\delta^q\|_1} - \frac{p^q}{\|p^q\|_1}\right)\right| = \frac{\delta^q}{m^{q-1}\|p_\delta^q\|_1}\left|-\frac{1}{n}\sum_{i=1}^{n} A_i + \frac{1}{m}\sum_{i=n+1}^{n+m} A_i\right|$$

$$= \frac{\delta^q}{(1-\delta)^q\left(\frac{n}{m}\right)^{1-q} + \delta^q}\left|-\frac{1}{n}\sum_{i=1}^{n} A_i + \frac{1}{m}\sum_{i=n+1}^{n+m} A_i\right|. \tag{6}$$

This expression is convergent as m goes to infinity with the following limit:

$$L := \left|-\frac{1}{n}\sum_{i=1}^{n} A_i + \bar{A}_{(n)}\right|,$$

where $\bar{A}_{(n)} = \lim_{m\to\infty} \frac{1}{m}\sum_{i=n+1}^{n+m} A_i$. If L is not zero - and it is not zero for most A-s - then we can choose an m so that (6) is greater than $L/2$. Therefore we have proved, that

Proposition 3. If $q < 1$, then the q-expectation value of $A \in l^\infty$, $D_q \to \mathbb{R}$, $p \mapsto (A \mid p^q/\|p^q\|)$ is not continuous if A satisfies is an $n \in \mathbb{N}$ so that

$$\left|-\frac{1}{n}\sum_{i=1}^{n} A_i + \lim_{m\to\infty}\frac{1}{m}\sum_{i=n+1}^{n+m} A_i\right| \neq 0.$$

■

Note that a number of A-s satisfy this condition.

3. Lesche stability and continuity

The original mathematical formulation of experimental robustness by Lesche is not continuity, but a related notion. He introduced "normalized" values of the corresponding functions instead of the "bare" values in the above definition of continuity [11, 14]. To clarify the relation of Lesche stability and continuity we introduce some additional notions. Let us see then the following sets

$$V_n := \{p \in D \mid p_i = 0 \text{ if } \ i > n\} \qquad n \in \mathbb{N},$$

$$V := \bigcup_n V_n.$$

It is clear that $V_m \in V_n$ if $m \leq n$. If $p \in V$ then let us define

$$n_p := \min\{n \in \mathbb{N} \mid p \in V_n\}.$$

Hence $p_i = 0$ if $i > n_p$.

Let $f : V \to \mathbb{R}$, $f \neq 0$ be a function with the property

$$\kappa_n := \sup\{|f(p)| \mid p \in V_n\} < \infty \qquad (n \in \mathbb{N}).$$

It is evident, that $\kappa_m \leq \kappa_n$, if $m \leq n$. Moreover, there is an $n_0 \in \mathbb{N}$ so that $\kappa_{n_0} > 0$.

Definition 2: A function f with the previous properties is Lesche-stable, if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall n > n_0)(\forall r, p \in V_n)\left(\|r-p\| < \delta \Rightarrow \frac{|f(r)-f(p)|}{\kappa_n} < \epsilon \right),$$

or equivalently

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall r, p \in V)\left(\|r-p\| < \delta \Rightarrow \frac{|f(r)-f(p)|}{\kappa_n} < \epsilon \right),$$

where $n := \max\{n_q, n_p\} > n_0$.

This definition corresponds to Lesche's original formulation [1].

Comparing the definitions of continuity and Lesche-stability it is clear, that

(1) If f is uniformly continuous, then it is Lesche-stable.
(2) If f is bounded and Lesche-stable, then it is uniformly continuous.

Lesche stability is a *global* property, However, the physical meaning of experimental robustness requires a refinement which is a *local* property. For example let us see the intuitive formulation of experimental robustness of Abe [16]:

> "Given a statistical mechanical system, perform a measurement to obtain a probability distribution $\{p_i\}_{i=1,\ldots,w}$ ... . Perform a measurement again on the same system prepared in the same state as before. Then another probability distribution $\{p'_i\}_{i=1,\ldots,w}$ will be obtained."

Continuing Abe requires that some related physical quantities do not be very different.

This formulation indicates that we want that in the neighbourhood of an *arbitrarily given* state the related physics do not change dramatically. The uniformity does not seem to be important.

Therefore we introduce the following concept of stability.

Definition 3: A function f is *stable at* $p \in V$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall r \in V \text{ and } n_r > n_0)\left(\|r-p\| < \delta \Rightarrow \frac{|f(r)-f(p)|}{\kappa_{n_r}} < \epsilon \right).$$

A function f is *stable* if it is stable at all states of its domain. Lesche-stability is uniform stability.

It is easy to see, that

(1) If f is continuous in p, then it is stable there.
(2) If f is bounded and stable in p, then it is continuous there.
(3) If f is Lesche-stable, then it is stable everywhere.
(4) If f is stable in a compact set of its domain, then it is Lesche-stable there.
(5) If f is instable then it is also Lesche-instable.

4. The stability of Rényi and Tsallis entropies and q-expectation values

4.1. **1<q.** We have shown in section 2.1, that the Rényi and Tsallis entropies are everywhere continuous, therefore they are stable.

We have also seen that the q-expectation value of a physical quantity $A \in l^\infty$ is continuous everywhere, therefore the q-expectation value is stable.

If $A \notin l^\infty$, then the q-expectation value is not necessarily continuous, however, it is stable.

In this case

$$\kappa_n = \sup\left\{ \left| \sum_{i=1}^{n} A_i \frac{p_i^q}{\|p^q\|} \right| \mid p \in V_n \right\} = \max_{i \le n} |A_i|,$$

therefore, if $n := \max\{n_r, n_p\} > n_0$, then

$$\frac{\left|\sum_{i=1}^n A_i \left(\frac{r_i^q}{\|r^q\|} - \frac{p_i^q}{\|p^q\|}\right)\right|}{\kappa_{n_r}} \le \frac{\kappa_{n_p}}{\kappa_{n_0}} \sum_{i=1}^{n} \left| \frac{r_i^q}{\|r^q\|} - \frac{p_i^q}{\|p^q\|} \right|.$$

The second term at the right hand side of this inequality is the difference of the continuous function $p \to p^q/\|p^q\|_1$ at values p and r, as we have seen in 2.1. Therefore, the right hand side of this inequality is smaller than ϵ choosing an r closer to p than $\delta = \kappa_{n_0}/\kappa_{n_p}\epsilon$.

4.2. **q<1.** In subsection 2.2 we have seen the Rényi and Tsallis entropies and the q-expectation values are not continuous, now we will show that they are not stable.

We can check that by a simple modification of the proofs in 2.2.

Considering p in (3) and $r = r_\delta$ in (4) we get for the Rényi entropy, that

$$\kappa_{n_r}^{\text{Rényi}} = \log(n+m)$$

and therefore the stability criteria is

$$\frac{|S_R(r) - S_R(p)|}{\kappa_{n_r}^{\text{Rényi}}} = \frac{\log((1-\delta)^q n^{1-q} + \delta^q m^{1-q}) - \log n^{1-q}}{\log(n+m)}. \tag{7}$$

This expression converges to $1-q$ as m goes to infinity. Therefore the Rényi entropy is instable.

Similarly for the Tsallis entropy we get

$$\kappa_{n_r}^{\text{Tsallis}} = \frac{n^{1-q} + m^{1-q} - 1}{1-q},$$

and the stability criteria is

$$\frac{|S_T(r) - S_T(p)|}{\kappa_{n_r}^{\text{Tsallis}}} = (1-q)\frac{|(1-(1-\delta)^q)n^{1-q} - \delta^q m^{1-q}|}{n^{1-q} + m^{1-q} - 1}. \tag{8}$$

This expression is convergent as m goes to infinity and has the limit

$$0 < L_T = \frac{1-q}{1-n^{q-1}}(1-(1-\delta)^q).$$

Therefore choosing m so that (8) be greater than $L_T/2$, we see that the Tsallis entropy is instable.

Regarding the stability of q-expectation values, it is enough to investigate only the case $A \in l^\infty$. Now

$$\kappa_{n_r}^{\text{q-av.}} \le \|A\|_\infty,$$

therefore the expression (6) divided by $\|A\|_\infty$ estimates the corresponding expression of the stability criteria. If A has the property given in Proposition 3 then the q-averages are instable.

5. Discussion

We have investigated some possible mathematical formulations of the experimental robustness of some physical quantities. The analysis of continuity, uniform continuity, and Lesche-stability revealed that these notions are closely related and it is convenient to introduce to use a local stability concept instead of the uniform notion of Lesche-stability. These formulations give essentially the same conditions of experimental robustness for the investigated functions:

The Rényi and Tsallis entropies are continuous and stable if $1 < q$ and are not continuous and instable for finite uniform distributions, if $q < 1$.

The q-expectation values are continuous and stable if $A \in l^\infty$ and $1 < q$ and are not necessarily continuous but stable if $A \notin l^\infty$ and $1 < q$. The q-expectation values are not continuous and instable for practically all physical quantities $A \in l^\infty$ (see the condition in 2.2) in case of finite uniform distributions.

Observe that the proof of Lesche [1] and Abe [16] for Lesche instability in the case in the case $1 < q$ does not negate our stability because they do not consider a neighbourhood of a *given* distribution (e.g. formula (7) in [16]) but a sequence of finite distributions whose length goes to infinity. The proof of Abe works in the case $q < 1$ but it shows the instability only for a single distribution.

If f is stable on a compact set of its domain, then it is also Lesche-stable. If f is instable then it is also Lesche-instable.

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